Problem Set IV: Due Tuesday, March 6, 2012

1a.) Using the same type of consideration as discussed in class, show that the decorrelation rate for a particle in an ensemble of stochastic electric fields in 1D scales as:

$$1/\tau_c \sim \left(k^2 D_{QL}\right)^{1/3}.$$

Here, $D_{QL}$ is the quasilinear diffusion co-efficient and decorrelation is defined relative to a wave number $k$.

b.) Extend the idea of integration over unperturbed orbits from 218A to show that formally the response of the distribution function to the electric field is given by

$$f_{\substack{k\\ \omega}} = -\frac{q}{m} E_{\substack{k\\ \omega}} \partial\langle f\rangle/\partial v \int_0^\infty \mathrm{d}\tau\, e^{i(\omega-kv)\tau} e^{ik\delta x(-\tau)},$$

where $\delta x(-\tau)$ is the deviation from the unperturbed obit.

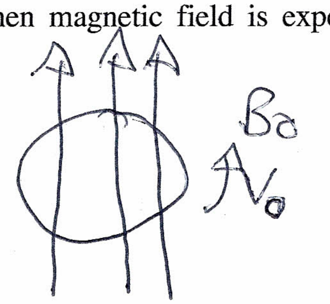
c.) Now, assume $\delta x(-\tau)$ is produced by diffusion in velocity and thus show

$$f_{\substack{k\\ \omega}} = -\frac{q}{m} E_{\substack{k\\ \omega}} \frac{\partial\langle f\rangle}{\partial v} \int_0^\infty \mathrm{d}\tau\, e^{i(\omega-kv)\tau} e^{-k^2 D\tau^3/3}.$$

d.) What is the physics of the effect discussed in c.)?

e.) Taking $\tau^3/\tau_c^3 \rightarrow \tau/\tau_c$ for convenience, what does this problem imply about resonance widths at finite amplitude? What does validity of the quasilinear calculation of $D$ imply about the resonance width?

2a.) What is the width of the magnetic boundary layer formed at the boundary of an eddy when magnetic field is expelled, in the case shown below (discussed in class).



b.) Will flux expulsion *always* occur? Estimate the conditions under which it will *not*. Discuss the physics of your result.

3.) Consider a bounded, 2D shear flow with $V_y = V_y(x)$, with no slip boundary conditions.

a.) Derive the linear equation governing the inviscid stability of this flow. (Hint: consider vorticity dynamics.)

b.) Show that an inflection point (i.e. a point where $V_y''(x) = 0$) is necessary for instability. (Hint: Assume growth, and construct a complex quadratic form. What does this form imply about the growth rate?)

c.) Derive the quasilinear equation for mean vorticity $\omega$ for this problem. When is this applicable?

d.) Now, assume that fluctuations are maintained by external stirring. What can be said about the mean time asymptotic vorticity profile assuming that no slip boundary conditions apply at each boundary? Discuss the physics of your result. (Hint: Review the H-theorem for quasilinear theory.)

4.) Consider the separation between two 'test' particles, attached to an ensemble of frozen field lines, with a fluctuation spectrum as predicted by the Goldreich-Sridhar model. Assume critical balance.

a.) Calculate the rate of separation of the neighboring particles, as they travel in the mean field ($\hat{z}$) direction. (Hint: Think Richardson, but for field lines.)

b.) How does the rate of separation compare to that of test particles in Navier-Stokes turbulence? What is the reason for this result?

5.a.) Consider a passive scalar, with concentration $c$, immersed in a turbulent flow. $c$ obeys the equation:

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla} c - D\nabla^2 = \tilde{f}_c .$$

Let $c$ have dissipation rate $\alpha$, i.e.

$$\alpha = \tilde{c}_0^2 \, \mathrm{v}_0 / \ell_0 .$$

i.) Calculate the $K41$ inertial range spectrum for concentration fluctuations.

ii.) *Quantitatively* discuss what happens if

$$D << v \text{ (viscosity)}, \; D >> v .$$

b.) Consider low Mach number incompressible turbulence, with $M = \mathrm{v}_0 / \mathrm{c_s} << 1$. Show

$$\ell_{diss} / \ell_{mfp} \sim M^{-1} Re^{1/4} .$$

Thus, the validity of continuum hydrodynamics gets *better* at high *Re.*